

On discrete series subrepresentations of generalized principle series

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Abstract

In this paper we study certain family of generalized principal series and obtain necessary and sufficient conditions under which such induced representations contain discrete series subquotients. Further, we show that if the generalized principal series which belongs to the studied family has a discrete series subquotient then it has a discrete series subrepresentation.

1 Introduction

Generalized principle series present a particularly interesting class of induced representations of classical p -adic groups, whose composition series have the main application in the determination of the unitary duals. These are the representations of the form $\delta \rtimes \sigma$, induced from the maximal parabolic subgroup from the representation having an irreducible essentially square-integrable representation δ on general linear part and an irreducible square-integrable (i.e., a discrete series) representation σ on the classical part. Non-unitary generalized principal series have been studied by Muić and reducibility of such representations has been fully described in [13] in terms of the classification of discrete series of classical p -adic groups by Mœglin and Tadić ([8], [10]). This important and essentially combinatorial classification, whose integral part are unitary generalized principal series (i.e., for δ unitary), now

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holds unconditionally due to results of [1]. In majority of cases, reducibility results are obtained identifying a non-tempered subquotient different than the Langlands one. It remains to obtain a deeper knowledge on the remaining part of composition series of generalized principal series, which can be of arbitrary rank, and it is of particular interest to determine the discrete series subquotients.

On the other hand, in case when σ is a strongly positive representation complete composition series of generalized principal series $\delta \rtimes \sigma$ are obtained in a uniform way in [12]. Besides other results, it has proved that if such generalized principal series contains a discrete series subquotient then it also contains a discrete series subrepresentation (if we write $\delta \rtimes \sigma$ as a standard representation, note that composition series of $\delta \rtimes \sigma$ and $\tilde{\delta} \rtimes \sigma$ coincide). Furthermore, it follows that, for strongly positive discrete series σ , every discrete series subquotient of the standard representation $\delta \rtimes \sigma$ has to be a subrepresentation, and such induced representation can have at most two discrete series subrepresentations, which appear only as an inductive step in Mœglin-Tadić classification.

The aim of this paper is to generalize mentioned subrepresentation results to other class of generalized principal series. In Mœglin-Tadić classification strongly positive representations serve as a basic building blocks for the construction of all discrete series, which are obtained adding in certain way two new consecutive elements in the Jordan block repeatedly. Thus, it is natural to start our investigation by considering a discrete series σ which is obtained by adding two consecutive elements in the Jordan block of a strongly positive representation. This provides a first inductive step in the construction of discrete series and provides much more complicated representations than the strongly positive ones. For instance, such representations can not be distinguished only by their Jordan blocks, unlike strongly positive ones. Among other applications, these representations have also played a key role in the determination of the first occurrence indices for theta lifts of discrete series of metaplectic groups in [5].

In the paper we are concerned with discrete series subquotients of so-called positive generalized principal series. This is the class of standard representations $\delta \rtimes \sigma$ where both σ and possible discrete series subquotient of generalized principal series $\delta \rtimes \sigma$ can be described as irreducible square-integrable representations obtained by adding two consecutive elements in the Jordan block of a strongly positive representation, no matter what the

cuspidal support of σ looks like.

Using Jacquet modules method, based on the structural formula of Tadić (a version of Geometrical lemma due to Bernstein and Zelevinsky [2]), we deduce necessary and sufficient conditions under which a positive generalized principal series contains a discrete series subquotient. Then for each positive generalized principal series that contains a discrete series subquotient we identify a discrete series subrepresentation. To achieve this, we again rely on Jacquet modules method, enhanced by intertwining operators method. Further, we examine different embeddings of discrete series which are investigated by precise calculation of certain Jacquet modules of some non-tempered irreducible representations. Such representations are observed from the point of known composition series in which they appear. In certain cases we derive discrete series subquotients of positive generalized principal series which do not appear as subrepresentations.

One of the main advantages of discrete series obtained by adding two consecutive elements in the Jordan block of a strongly positive representation is that substantial set of their Jacquet modules can be deduced applying the structural formula of Tadić to known embeddings of such representations. Using a complete description of Jacquet modules of strongly positive representations given in [6], which follows from an algebraic classification of such representations ([4]), classical group parts of Jacquet modules of investigated representations can be deduced using [12]. An uniform description of such Jacquet modules can be found in [7].

We briefly describe the content of the paper, section by section.

In the following section we present some preliminaries and introduce the notion of positive generalized principle series. In Section 3 we prove two technical results which are used several times through the paper. In the fourth section we obtain necessary and sufficient conditions under which positive generalized principal series contains a discrete series subquotient, while in the fifth section for each such induced representation we explicitly construct its discrete series subrepresentation.

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2 Notation and preliminaries

Let F denote a non-archimedean local field of characteristic different from two. In this paper we consider usual towers of symplectic and orthogonal groups $G_n = G(V_n)$, that are the groups of isometries of F -spaces $(V_n, (\ , \))$, $n \geq 0$. Here the form $(\ , \)$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric if the tower is orthogonal. The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F -parabolic subgroup in the classical group G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M \simeq GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes a general linear group of rank m over F . If δ_i is a representation of $GL(n_i, F)$ and τ a representation of $G_{n'}$, a normalized parabolically induced representation $Ind_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$. We use similar notation to denote a parabolically induced representation of $GL(m, F)$.

By $Irr(G_n)$ we denote the set of all irreducible admissible representations of G_n . Further, let $R(G_n)$ denote a Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \bigoplus_{n \geq 0} R(G_n)$. In a similar way we define $R(GL) = \bigoplus_{n \geq 0} R(GL(n, F))$. For $\sigma \in Irr(G_n)$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having Levi subgroup equal $GL(k, F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F)) \otimes R(G_{n-k})$. We can consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^n r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

The following lemma, which has been derived in [16], presents a crucial structural formula for our calculations with Jacquet modules.

Lemma 2.1. *Let ρ be an irreducible cuspidal representation of $GL(m, F)$ and $k, l \in \mathbb{R}$ such that $a + b \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of G_n . Write $\mu^*(\sigma) = \sum_{\tau, \sigma'} \tau \otimes \sigma'$. Then the following holds:*

$$\begin{aligned} \mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) &= \sum_{i=-k-1}^l \sum_{j=i}^l \sum_{\tau, \sigma'} \delta([\nu^{-i}\tilde{\rho}, \nu^k\tilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \tau \\ &\otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'. \end{aligned}$$

We omit $\delta([\nu^x \rho, \nu^y \rho])$ if $x > y$.

We will also fix the notation that will be used through the paper.

The induced representation of the form $\delta \rtimes \sigma$, where δ is an irreducible essentially square integrable representation of general linear group and σ is a discrete series representation of G_n , is called a generalized principle series.

There is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)} \delta$ is unitarizable, where ν denotes a p -adic norm of the field F . If $e(\delta) > 0$, generalized principle series $\delta \rtimes \sigma$ has a unique irreducible (Langlands) quotient, which is the unique irreducible subrepresentation of $\tilde{\delta} \rtimes \sigma$ ($\tilde{\delta}$ denotes the contragradient of δ).

By the results of [18], the representation δ is attached to the segment and we write $\delta = \delta([\nu^a \rho, \nu^b \rho])$, where $a, b \in \mathbb{R}$ such that $b - a$ is a nonnegative integer and ρ is an irreducible unitary representation of $GL(n_\rho, F)$. We recall that $\delta([\nu^a \rho, \nu^b \rho])$ is a unique irreducible subrepresentation of the induced representation $\nu^b \rho \times \nu^{b-1} \rho \times \cdots \times \nu^a \rho$.

Throughout the paper we prefer to use a subrepresentation version of Langlands classification and write a non-tempered irreducible representation π of G_n as a unique irreducible (Langlands) subrepresentation of the induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is a tempered representation of G_t , δ_i is an irreducible essentially square integrable representation of $GL(n_{\delta_i}, F)$ attached to the segment $[\nu^{a_i} \rho_i, \nu^{b_i} \rho_i]$ for $i = 1, 2, \dots, k$, and $a_1 + b_1 \leq a_2 + b_2 \leq \cdots \leq a_k + b_k < 0$. In this case, we also write $\pi = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$.

We briefly recollect of Mœglin-Tadić classification of discrete series for groups that we consider. We fix a certain tower of classical groups (symplectic or special odd orthogonal). Every discrete series representation is uniquely described by three invariants: a partial cuspidal support, Jordan block and ϵ -function.

A partial cuspidal support of a discrete series $\sigma \in Irr(G_n)$ is an irreducible cuspidal representation σ_{cusp} of some G_m such that there is an irreducible admissible representation π of $GL(n_\pi, F)$ such that σ is a subrepresentation of $\pi \rtimes \sigma_{cusp}$.

Jordan block of σ , denoted by $Jord(\sigma)$, is a set of all pairs (c, ρ) where $\rho \simeq \tilde{\rho}$ is an irreducible cuspidal representation of some $GL(n_\rho, F)$ and $c > 0$ is an integer such that the following two conditions are satisfied:

1. c is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L -function $L(s, \rho, r)$ is the one defined by Shahidi (see for instance [14],

[15]), where $r = \bigwedge^2 \mathbb{C}^{n_\rho}$ is the exterior square representation of the standard representation on \mathbb{C}^{n_ρ} of $GL(n_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and $r = \text{Sym}^2 \mathbb{C}^{n_\rho}$ is the symmetric-square representation of the standard representation on \mathbb{C}^{n_ρ} of $GL(n_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

2. The induced representation

$$\delta([\nu^{-(a-1)/2} \rho, \nu^{(a-1)/2} \rho]) \rtimes \sigma$$

is irreducible.

To explain the notion of the ϵ -function, we will first define Jordan triples. This are the triples of the form $(Jord, \sigma', \epsilon)$ where

- σ' is a cuspidal representation of some G_n .
- $Jord$ is a finite set (possibly empty) of pairs (c, ρ) ($\rho \simeq \tilde{\rho}$ is an irreducible cuspidal representation of $GL(n_\rho, F)$, $c > 0$ an integer) such that c is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$ (as above). For an irreducible cuspidal representation $\rho \simeq \tilde{\rho}$ of $GL(n_\rho, F)$ we write $Jord_\rho = \{c : (c, \rho) \in Jord\}$. If $Jord_\rho \neq \emptyset$ and $c \in Jord_\rho$, we put $c_- = \max\{d \in Jord_\rho : d < c\}$, if it exists.
- ϵ is a function defined on a subset of $Jord \cup (Jord \times Jord)$ and attains values 1 and -1. If $(c, \rho) \in Jord$, then $\epsilon(c, \rho)$ is not defined if and only if c is odd and $(c', \rho) \in Jord(\sigma')$ for some positive integer c' . Next, ϵ is defined on a pair $(c, \rho), (c', \rho') \in Jord$ if and only if $\rho \simeq \rho'$ and $c \neq c'$.

The following compatibility conditions must hold for different $c, c', c'' \in Jord_\rho$:

1. If $\epsilon(c, \rho)$ is defined (hence $\epsilon(c', \rho)$ is also defined), then $\epsilon((c, \rho), (c', \rho)) = \epsilon(c, \rho) \cdot \epsilon(c', \rho)^{-1}$.
2. $\epsilon((c, \rho), (c'', \rho)) = \epsilon((c, \rho), (c', \rho)) \cdot \epsilon((c', \rho), (c'', \rho))$.
3. $\epsilon((c, \rho), (c', \rho)) = \epsilon((c', \rho), (c, \rho))$.

Listed properties show that it is enough to know the value of ϵ on the consecutive pairs $\epsilon((c_-, \rho), (c, \rho))$ and on the minimal element of $Jord_\rho$ (if it is defined on elements, not only on pairs).

Suppose that, for Jordan triple $(Jord, \sigma', \epsilon)$, there is $(c, \rho) \in Jord$ such that $\epsilon((c_-, \rho), (c, \rho)) = 1$. If we put $Jord' = Jord \setminus \{(c_-, \rho), (c, \rho)\}$ and consider the restriction ϵ' of ϵ to $Jord' \cup (Jord' \times Jord')$, we obtain a new Jordan triple $(Jord', \sigma', \epsilon')$, and we say that such Jordan triple is subordinated to $(Jord, \sigma', \epsilon)$.

We say that Jordan triple $(Jord, \sigma', \epsilon)$ is a triple of alternated type if $\epsilon((c_-, \rho), (c, \rho)) = -1$ holds whenever c_- is defined and there is an increasing bijection $\phi_\rho : Jord_\rho \rightarrow Jord'_\rho(\sigma')$, where $Jord'_\rho(\sigma')$ equals $Jord_\rho(\sigma') \cup \{0\}$ if a is even and $\epsilon(\min Jord_\rho, \rho) = 1$ and $Jord'_\rho(\sigma')$ equals $Jord_\rho(\sigma')$ otherwise.

Jordan triple $(Jord, \sigma', \epsilon)$ dominates the Jordan triple $(Jord', \sigma', \epsilon')$ if there is a sequence of Jordan triples $(Jord_i, \sigma', \epsilon_i)$, $0 \leq i \leq k$, such that $(Jord_0, \sigma', \epsilon_0) = (Jord, \sigma', \epsilon)$, $(Jord_k, \sigma', \epsilon_k) = (Jord', \sigma', \epsilon')$ and $(Jord_i, \sigma', \epsilon_i)$ is subordinated to $(Jord_{i-1}, \sigma', \epsilon_{i-1})$ for $i \in \{1, 2, \dots, k\}$. Jordan triple $(Jord, \sigma', \epsilon)$ is called the admissible triple if it dominates a triple of alternated type.

The classification given in [8], [10] (which now holds without any assumptions, due to results of Arthur [1], some details can also be seen in [9]) states that there is one-to-one correspondence between the set of all discrete series in $Irr(G)$ and the set of all admissible triples $(Jord, \sigma', \epsilon)$ given by $\sigma = \sigma_{(Jord, \sigma', \epsilon)}$, such that $\sigma_{cusp} = \sigma'$ and $Jord(\sigma) = Jord$. Further, if $(c, \rho) \in Jord$ is such that $\epsilon((c_-, \rho), (c, \rho)) = 1$, we set $Jord' = Jord \setminus \{(c_-, \rho), (c, \rho)\}$ and consider the restriction ϵ' of ϵ to $Jord' \cup (Jord' \times Jord')$. Then $(Jord', \sigma', \epsilon')$ is an admissible triple and σ is a subrepresentation of $\delta([\nu^{-(c_- - 1)/2} \rho, \nu^{(c - 1)/2} \rho]) \rtimes \sigma_{(Jord', \sigma', \epsilon')}$. Such induced representation has exactly two discrete series subrepresentations, which are non-isomorphic. Moreover, the induced representation $\delta([\nu^{-(c_- - 1)/2} \rho, \nu^{(c - 1)/2} \rho]) \rtimes \sigma_{(Jord', \sigma', \epsilon')}$ is a direct sum of two non-isomorphic tempered representations τ_+ and τ_- and there is the unique $\tau \in \{\tau_+, \tau_-\}$ such that σ is a subrepresentation of $\delta([\nu^{(c_- + 1)/2} \rho, \nu^{(c - 1)/2} \rho]) \rtimes \tau$.

We shall also say that discrete series σ and its corresponding admissible triple $(Jord, \sigma', \epsilon)$ are attached to each other.

It has been shown in [8] and [10] that triples of alternated type correspond to strongly positive discrete series. Thus, the strongly positive discrete series serve as a cornerstone in described construction of discrete series and classifi-

cation of such representations of metaplectic groups, which also holds in the classical group case, is given in [4]. Definition of the triples of alternated type implies that such discrete series are completely determined by their partial cuspidal support and Jordan block. Since all discrete series which we study share a common partial cuspidal support we will define only Jordan block when introducing some strongly positive discrete series. Such procedure is also summarized in Proposition 1.2 of [12].

Further, here and subsequently we assume that σ is a discrete series representation of G_n and that there are $c_-, c \in \text{Jord}_{\rho'}(\sigma)$ such that σ is a subrepresentation of the induced representation of the form

$$\delta([\nu^{-(c_- - 1)/2} \rho', \nu^{(c - 1)/2} \rho']) \rtimes \sigma_{sp}$$

for strongly positive representation σ_{sp} such that $[c_-, c] \cap \text{Jord}_{\rho'}(\sigma_{sp}) = \emptyset$. In other words, if we put $\text{Jord}' = \text{Jord}(\sigma) \setminus \{(c_-, \rho'), (c, \rho')\}$ and denote by ϵ' the restriction of ϵ_σ to Jord' , then σ_{sp} corresponds to an admissible triple $(\text{Jord}', \sigma_{cusp}, \epsilon')$ of alternated type.

Obviously, $\epsilon((c_-, \rho), (c, \rho)) = 1$ and pair c_-, c with this property may not be unique. Since σ_{sp} is strongly positive, there can be at most two such pairs.

We are interested in deriving necessary and sufficient conditions under which the generalized principal series $\delta \rtimes \sigma$, with δ and σ as above, contains a discrete series subquotient. In particular, in this case the induced representation $\delta \rtimes \sigma$ reduces so we may assume $\text{Jord}_\rho \neq \emptyset$ and $a - c \in \mathbb{Z}$, $\forall 2c + 1 \in \text{Jord}_\rho$ (this has already been observed in [11] and also implies $\rho \simeq \tilde{\rho}$). Further, by the last case considered in Subsection 4.2 of [3], if $\delta \rtimes \sigma$ contains a discrete series subquotient then $2a - 1 \in \text{Jord}_\rho$ and $2b + 1 \notin \text{Jord}_\rho$.

To shorten notation, we introduce the following concept:

Definition. We call a generalized principle series $\delta \rtimes \sigma$ positive if the following conditions hold:

- $\delta = \delta([\nu^a \rho, \nu^b \rho])$, where $a, b \in \mathbb{R}$, $b - a \in \mathbb{Z}_{\geq 0}$, $a > \frac{1}{2}$,
- there are $c_-, c \in \text{Jord}_{\rho'}(\sigma)$ such that $\sigma \hookrightarrow \delta([\nu^{-(c_- - 1)/2} \rho', \nu^{(c - 1)/2} \rho']) \rtimes \sigma_{sp}$ for strongly positive representation σ_{sp} such that $[c_-, c] \cap \text{Jord}_{\rho'}(\sigma_{sp}) = \emptyset$,
- $\text{Jord}_\rho(\sigma) \neq \emptyset$ and $a - c \in \mathbb{Z}$, $\forall 2c + 1 \in \text{Jord}_\rho$,
- $2a - 1 \in \text{Jord}_\rho(\sigma)$ and $2b + 1 \notin \text{Jord}_\rho(\sigma)$.

3 Some technical results

This section is devoted to the proof of two technical results which will be frequently used afterwards in the paper.

Lemma 3.1. *Let $\sigma \in \text{Irr}(G_n)$ denote a discrete series representation and suppose that there are $c_-, c \in \text{Jord}_{\rho'}(\sigma)$ such that σ is a subrepresentation of $\delta([\nu^{-(c_-+1)/2}\rho', \nu^{(c-1)/2}\rho']) \rtimes \sigma_{sp}$ for strongly positive representation σ_{sp} such that $[c_-, c] \cap \text{Jord}_{\rho'}(\sigma_{sp}) = \emptyset$. If σ is written as a subrepresentation of the induced representation $\delta \rtimes \pi$, where δ is an essentially square-integrable representation of $GL(n_\delta, F)$ and π a non-tempered representation of the form $L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$, then $k = 1$.*

Proof. We write $\delta = \delta([\nu^x \rho, \nu^y \rho])$ and $\pi = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$ where $\delta_i = \delta([\nu^{a_i} \rho_i, \nu^{b_i} \rho_i])$ for $i = 1, 2, \dots, k$, and $a_1 + b_1 \leq a_2 + b_2 \leq \cdots \leq a_k + b_k < 0$. To obtain a contradiction, suppose that $k \geq 2$.

Since π is a subrepresentation of $\delta_1 \rtimes L(\delta_2 \times \cdots \times \delta_k \rtimes \tau)$, the square-integrability of σ shows that σ is contained in the kernel of an intertwining operator

$$\delta \times \delta_1 \rtimes L(\delta_2 \times \cdots \times \delta_k \rtimes \tau) \rightarrow \delta_1 \times \delta \rtimes L(\delta_2 \times \cdots \times \delta_k \rtimes \tau).$$

This also implies $\rho_1 \simeq \rho$. By [18], this kernel equals

$$\delta([\nu^{a_1} \rho, \nu^y \rho]) \times \delta([\nu^x \rho, \nu^{b_1} \rho]) \rtimes L(\delta_2 \times \cdots \times \delta_k \rtimes \tau).$$

Note that the representation $\delta([\nu^{a_1} \rho, \nu^y \rho]) \times \delta([\nu^x \rho, \nu^{b_1} \rho])$ is irreducible. Further, $L(\delta_2 \times \cdots \times \delta_k \rtimes \tau)$ is a subrepresentation of $\delta_2 \rtimes L(\delta_3 \times \cdots \times \delta_k \rtimes \tau)$. This gives us an embedding

$$\sigma \hookrightarrow \delta([\nu^{a_1} \rho, \nu^y \rho]) \times \delta([\nu^x \rho, \nu^{b_1} \rho]) \times \delta_2 \rtimes L(\delta_3 \times \cdots \times \delta_k \rtimes \tau).$$

Using Frobenius reciprocity, we obtain that Jacquet module of σ with respect to an appropriate parabolic subgroup contains $\delta([\nu^{a_1} \rho, \nu^y \rho]) \times \delta([\nu^x \rho, \nu^{b_1} \rho]) \otimes \delta_2 \otimes L(\delta_3 \times \cdots \times \delta_k \rtimes \tau)$.

Since σ is contained in $\delta([\nu^{-(c_-+1)/2}\rho', \nu^{(c-1)/2}\rho']) \rtimes \sigma_{sp}$, applying Lemma 2.1 we obtain that there are $-(c_-+1)/2 \leq i_1 \leq j_1 \leq (c-1)/2$ and $\pi_1 \otimes \sigma_1 \leq \mu^*(\sigma_{sp})$ such that

$$\delta([\nu^{a_1} \rho, \nu^y \rho]) \times \delta([\nu^x \rho, \nu^{b_1} \rho]) \leq \delta([\nu^{-i_1} \rho', \nu^{(c-1)/2} \rho']) \times \delta([\nu^{j_1+1} \rho', \nu^{(c-1)/2} \rho']) \times \pi_1$$

and

$$\delta_2 \otimes L(\delta_3 \times \cdots \times \delta_k \rtimes \tau) \leq \mu^*(\delta([\nu^{i_1+1}\rho', \nu^{j_1}\rho']) \rtimes \sigma_1). \quad (1)$$

Since σ_{sp} is strongly positive and $a_1 < 0$, Theorem 4.6 of [6] implies $\rho \simeq \rho'$. Further, either $-i_1 = a_1$ or $j_1 + 1 = a_1$ and, consequently, $i_1 \cdot j_1 > 0$. Observe that σ_1 is also strongly positive so (1) gives $i_1 < j_1$. We define $z = j_1$ if $i_1 > 0$ and $z = -i_1 - 1$ otherwise. It follows that $\delta_2 \otimes L(\delta_3 \times \cdots \times \delta_k \rtimes \tau) \leq \mu^*(\delta([\nu^{-a_1+1}\rho', \nu^z\rho']) \rtimes \sigma_1)$. Here we have used the fact that for every irreducible essentially square-integrable representation δ' holds $\mu^*(\delta' \rtimes \sigma_1) = \mu^*(\widetilde{\delta'} \rtimes \sigma_1)$. Repeating the same procedure, we obtain that there are $-a_1 \leq i_2 \leq j_2 \leq z$ and $\pi_2 \otimes \sigma_2 \leq \mu^*(\sigma_1)$ such that

$$\delta([\nu^{a_2}\rho_2, \nu^{b_2}\rho_2]) \leq \delta([\nu^{-i_2}\rho', \nu^{a_1-1}\rho']) \times \delta([\nu^{j_2+1}\rho', \nu^z\rho']) \times \pi_2.$$

Now $a_2 < 0$, $j_2 > 0$ and strong positivity of σ_1 lead to $\rho_2 \simeq \rho'$, $i_2 \geq -a_1 + 1$, $a_2 = -i_2$ and $b_2 = a_1 - 1$. This contradicts our assumption $a_1 + b_1 \leq a_2 + b_2$. Thus, $k = 1$. □

In the last section we will also need the following correction of the part (iv) of Theorem 4.1 from [12]. Let us shortly recall the required notation.

Let σ_{sp} denote a strongly positive discrete series of G_n , attached to the admissible triple of alternated type $(Jord, \sigma_{cusp}, \epsilon)$. We suppose that δ is an irreducible essentially square integrable representation of $GL(n_\rho, F)$ attached to the segment $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho])$ such that $Jord_\rho \neq \emptyset$ and $l_1 - c \in \mathbb{Z}$, $\forall 2c + 1 \in Jord_\rho$. Further, we assume $\min(Jord_\rho \cap [2l_1 + 1, 2l_2 + 1]) = \max(Jord_\rho \cap [2l_1 + 1, 2l_2 + 1])$ and denote $\max(Jord_\rho \cap [2l_1 + 1, 2l_2 + 1])$ by $2c + 1$. Also, we introduce strongly positive discrete series σ_1 and σ_2 such that $Jord(\sigma_1) = Jord \setminus \{(2c+1, \rho)\} \cup \{(2l_1+1, \rho)\}$ and $Jord(\sigma_2) = Jord \setminus \{(2c+1, \rho)\} \cup \{(2l_2+1, \rho)\}$. Finally, we define discrete series σ_{ds} whose corresponding admissible triple is $(Jord', \sigma_{cusp}, \epsilon')$ such that $Jord' = Jord \cup \{(2l_1+1, \rho), (2l_2+1, \rho)\}$ and $\epsilon'((2c+1, \rho), (2l_2+1, \rho)) = \epsilon'((2l_1+1, \rho), (2c+1, \rho)) = 1$, while on all other pairs ϵ' equals -1 . The following result will be very important for our considerations.

Proposition 3.2. *In an appropriate Grothendieck group we have*

$$\delta \rtimes \sigma = L(\widetilde{\delta} \rtimes \sigma) + L(\delta([\nu^{-l_2}\rho, \nu^c\rho]) \rtimes \sigma_1) + L(\delta([\nu^{-c}\rho, \nu^{l_1}\rho]) \rtimes \sigma_2) + \sigma_{ds}.$$

Proof. It suffices to show that σ_{ds} is the only tempered subquotient of $\delta \rtimes \sigma$. First we show that σ_{ds} is a subrepresentation of $\delta \rtimes \sigma$. Condition $\epsilon'((2c+1, \rho), (2l_2+1, \rho)) = 1$ gives the following embeddings:

$$\sigma_{ds} \hookrightarrow \delta([\nu^{-c}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1.$$

From Lemma 3.2 of [10], we deduce that there is some irreducible representation τ such that σ_{ds} is a subrepresentation of $\delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau$. Frobenius reciprocity forces $\mu^*(\sigma_{ds}) \geq \delta \otimes \tau$. We will show that τ has to be a discrete series representation. Suppose, on the contrary, that τ is not a discrete series. Then Section 8 from [10] shows that τ is a non-tempered representation and, using previous lemma, we realize τ as $L(\delta_1 \rtimes \tau_t)$. Since $\delta_1 = \delta([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1])$ and $a_1 + b_1 < 0$, from cuspidal support of σ_{ds} we obtain $b_1 < 0$ and $\rho_1 \simeq \rho$. In the same way as in the proof of previous lemma we deduce that σ_{ds} is contained in

$$\delta([\nu^{a_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{b_1}\rho]) \rtimes \tau_t.$$

Since $\delta([\nu^{a_1}\rho, \nu^{l_2}\rho]) \times \delta([\nu^{-l_1}\rho, \nu^{b_1}\rho]) \simeq \delta([\nu^{-l_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{l_2}\rho])$, we obtain $b_1 = -l_1 - 1$ and $\sigma_{ds} \hookrightarrow \delta([\nu^{a_1}\rho, \nu^{l_2}\rho]) \rtimes \tau_t$. This also gives $l_1 < -a_1 < l_2$.

Since σ_{ds} is contained in $\delta([\nu^{-c}\rho, \nu^{l_2}\rho]) \rtimes \sigma_1$, applying Lemma 2.1 we obtain that there are $-c - 1 \leq i_1 \leq j_1 \leq l_2$ and $\pi_1 \otimes \sigma'_1 \leq \mu^*(\sigma_1)$ such that

$$\delta([\nu^{a_1}\rho, \nu^{l_2}\rho]) \leq \delta([\nu^{-i_1}\rho, \nu^c\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^{l_2}\rho]) \times \pi_1$$

and

$$\tau_t \leq \mu^*(\delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \rtimes \sigma'_1).$$

Since σ_1 is strongly positive, Theorem 4.6 of [6] implies $(i_1, j_1) \in \{(-c-1, a_1-1), (-a_1, c-1)\}$. This gives $\tau_t \leq \delta([\nu^{-c}\rho, \nu^{a_1-1}\rho]) \rtimes \sigma_1$. Cuspidal support of σ_{ds} implies that τ_t is strongly positive and from [12], Proposition 3.1, we get $a_1 = -c$. Thus, if τ is non-tempered then $\tau \simeq L(\delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1)$.

On the other hand, condition $\epsilon'((2l_1+1, \rho), (2c+1, \rho)) = 1$ implies $\mu^*(\sigma_{ds}) \geq \delta([\nu^{l_1+1}\rho, \nu^c\rho]) \otimes \tau'$ for some irreducible representation τ' . Since $\sigma_{ds} \hookrightarrow \delta([\nu^{-l_1}\rho, \nu^{l_2}\rho]) \rtimes \tau$ and $l_2 > c$, it is not hard to obtain $\mu^*(\tau) \geq \delta([\nu^{l_1+1}\rho, \nu^c\rho]) \otimes \tau''$ for some irreducible representation τ'' .

By Proposition 3.1 of [12], we have

$$\delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1 = L(\delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1) + \sigma.$$

Results of [6] imply $\mu^*(\sigma) \geq \delta([\nu^{l_1+1}\rho, \nu^c\rho]) \otimes \sigma_1$. Further, it follows directly from definition of σ_1 and Lemma 2.1 that $\delta([\nu^{l_1+1}\rho, \nu^c\rho]) \otimes \sigma_1$ is the only representation of the form $\delta([\nu^{l_1+1}\rho, \nu^c\rho]) \otimes \tau''$ appearing in $\mu^*(\delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1)$. Therefore, σ_{ds} is not contained in $\delta \rtimes L(\delta([\nu^{-c}\rho, \nu^{-l_1-1}\rho]) \rtimes \sigma_1)$.

This enables us to conclude that τ is discrete series representation. Further, from cuspidal support of σ_{ds} we see that τ is strongly positive. Lemma 3.6 of [6] gives $\tau \simeq \sigma$, i.e., $\sigma_{ds} \hookrightarrow \delta \rtimes \sigma$.

It remains to prove that σ_{ds} is the only irreducible tempered subquotient of $\delta \rtimes \sigma$.

Suppose, contrary to our claim, that there is some other tempered subquotient of $\delta \rtimes \sigma$, we denote such subquotient by τ . Section 8 of [10] shows that τ is in discrete series and examining the cuspidal support of τ we deduce at once that it is not strongly positive. We assume that τ corresponds to admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$, where $Jord^{(1)} = Jord \cup \{(2l_1 + 1, \rho), (2l_2 + 1, \rho)\}$. Since τ is not strongly positive, there is some $(d, \rho') \in Jord^{(1)}$ such that $\epsilon'((d_-, \rho'), (d, \rho')) = 1$. In the same way as in Section 4 of [12] we obtain $\rho' \simeq \rho$ and $d \in \{2c + 1, 2l_2 + 1\}$. If $d = 2c + 1$, we obtain from [10] that $\mu^*(\tau) \geq \delta([\nu^{-l_1}\rho, \nu^c\rho]) \otimes \sigma_2$, while if $d = 2l_2 + 1$ we have $\mu^*(\tau) \geq \delta([\nu^{-c}\rho, \nu^{l_2}\rho]) \otimes \sigma_1$. But, it can be easily verified that both these representation appear with multiplicity one in $\mu^*(\delta \rtimes \sigma)$ and they also appear in $\mu^*(\sigma_{ds})$. Consequently, such representation do not appear in $\mu^*(\tau)$.

This contradicts our assumption and proves the proposition. \square

4 Discrete series subquotients

In this section we will study under which conditions a discrete series representation appears in the composition series of positive generalized discrete series $\delta \rtimes \sigma$. Here and subsequently, we suppose that σ is attached to the admissible triple $(Jord, \sigma_{cusp}, \epsilon)$.

Obviously, if σ_{ds} is a discrete series subquotient of $\delta \rtimes \sigma$, then σ_{ds} corresponds to the admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$, where $Jord^{(1)}$ equals $Jord \setminus \{(\rho, 2a - 1)\} \cup \{(\rho, 2b + 1)\}$. Also, inspecting the cuspidal support of $\delta \rtimes \sigma$ we deduce at once that there is $d \in Jord_\rho^{(1)}$ such that σ_{ds} is a subrepresentation of

$$\delta([\nu^{-(d-1)/2}\rho'', \nu^{(d-1)/2}\rho'']) \rtimes \sigma_{sp}^{(1)},$$

for some strongly positive discrete series $\sigma_{sp}^{(1)}$.

We start from the case which follows directly from [17], Theorem 8.2.

Lemma 4.1. *If $Jord_\rho \cap [2a+1, 2b+1] = \emptyset$, then generalized principal series $\delta \rtimes \sigma$ contains a discrete series subquotient.*

The rest of this section will be devoted to the case $Jord_\rho \cap [2a+1, 2b+1] \neq \emptyset$. In the sequel, let c_{min} (resp., c_{max}) denote the minimum (resp., maximum) of the set $Jord_\rho \cap [2a+1, 2b+1]$. Note that $(c_{min})_- = 2a-1$. First we have an elementary, but important, technical result.

Lemma 4.2. *If positive generalized principal series $\delta \rtimes \sigma$ contains a discrete series subquotient and $Jord_\rho \cap [2a+1, 2b+1] \neq \emptyset$, then $\epsilon((2a-1, \rho), (c_{min}, \rho)) = 1$.*

Proof. Suppose that σ_{ds} is a discrete series subquotient of $\delta \rtimes \sigma$ which corresponds to admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$. Since $(\rho, 2a-1) \notin Jord^{(1)}$, Theorem 8.2 from [17] gives $\mu^*(\sigma_{ds}) \geq \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \otimes \pi$ for some irreducible representation π of $G_{n'}$. Since $(c_{min}-1)/2$ is less than b , using Lemma 2.1 one can easily deduce that $\mu^*(\sigma)$ contains $\delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \otimes \pi'$ for some irreducible representation π' and Proposition 7.2 of [17] yields $\epsilon((2a-1, \rho), (c_{min}, \rho)) = 1$. This proves the lemma. \square

In the following sequence of propositions, several possibilities will be studied separately.

Proposition 4.3. *If $c_{min} < (c_{max})_-$ then positive generalized principal series does not contain a discrete series subquotient.*

Proof. If $\epsilon((2a-1, \rho), (c_{min}, \rho)) = -1$, statement of the proposition follows from the previous lemma. Thus, we may assume $\epsilon((2a-1, \rho), (c_{min}, \rho)) = 1$. This also shows $\rho \simeq \rho'$. We denote by c an element of $Jord_\rho$ such that $c_{min} = c_-$.

Two possibilities will be considered separately.

Let us first assume $(c_{max})_- \neq c$. Suppose on the contrary that σ_{ds} is a discrete series subquotient of $\delta \rtimes \sigma$, which corresponds to the admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$. There is $(d, \rho'') \in Jord^{(1)}$ such that $\epsilon^{(1)}((d_-, \rho''), (d, \rho'')) = 1$. This implies $\mu^*(\sigma_{ds}) \geq \delta([\nu^{-(d-1)/2} \rho'', \nu^{(d-1)/2} \rho'']) \otimes \sigma_{sp}^{(1)}$, for appropriate strongly positive discrete series $\sigma_{sp}^{(1)}$. Since σ_{ds} is an irreducible subquotient of σ , the same representation appears in $\mu^*(\delta \rtimes \sigma)$. Using structural formula

recalled in Lemma 2.1 we conclude that there are $a - 1 \leq i \leq j \leq b$ and an irreducible representation $\tau \otimes \sigma' \leq \mu^*(\sigma)$ such that

$$\delta([\nu^{-(d-1)/2}\rho'', \nu^{(d-1)/2}\rho'']) \leq \delta([\nu^{-i}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \tau.$$

Obviously, $\rho \simeq \rho''$. If $(d-1)/2 = b$ then we have $d_- = c_{\max}$. Further, since $2b+1 \notin \text{Jord}_\rho$, either $i = (c_{\max} - 1)/2$ and $\tau \simeq \delta([\nu^{-a+1}\rho, \nu^j\rho])$ or $\tau \simeq \delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^j\rho])$ holds.

In the first case, since $-a+1 \leq 0$ and $j \geq (c_{\max}+1)/2$, we obtain $i = j$ and transitivity of Jacquet modules shows $\mu^*(\sigma) \geq \delta([\nu^{(c_{\max}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \otimes \sigma''$, for some irreducible representation σ'' , which forces $\epsilon(((c_{\max})_-, \rho), (c_{\max}, \rho)) = 1$. But this is impossible, since $\epsilon((2a-1, \rho), (c_{\min}, \rho)) = 1$ and definition of positive generalized principle series implies $\epsilon((x_-, \rho), (x, \rho)) = -1$ for $x > c$.

In the second case, using a well-known fact that $\delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^j\rho]) \otimes \sigma' \leq \mu^*(\sigma)$ implies $2j+1 \in \text{Jord}_\rho$, we get a contradiction since square-integrability of σ implies $j > (c_{\max} - 1)/2$ and $j > b$.

Thus, $d \neq 2b+1$ and it follows directly from square-integrability of σ that τ is of the form $\delta([\nu^k\rho, \nu^{(d-1)/2}\rho])$, for some $k < 0$. This implies $d \in \text{Jord}_\rho \cap \text{Jord}_\rho^{(1)}$ and, similarly as in previously considered case, we obtain $\epsilon((x, \rho), (d, \rho)) = 1$, where x is an element of Jord_ρ such that $\langle x, d \rangle \cap \text{Jord}_\rho = \emptyset$. By definition of positive generalized discrete series and Lemma 4.2, either $d = c$ or $d = c_{\min}$. Note that $d = c$ forces $\epsilon((c_{\min}, \rho), (c, \rho)) = 1$, while $d = c_{\min}$ implies $2a-1 > \min(\text{Jord}_\rho)$. We will discuss only the case $d = c$, the other case can be handled in the same way but more easily.

Assumption $d = c$ implies $\epsilon^{(1)}((c_{\min}, \rho), (c, \rho)) = 1$. We have $\mu^*(\delta \rtimes \sigma) \geq \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma'_{sp}$, where σ'_{sp} is strongly positive discrete series such that $\text{Jord}_{\rho'}(\sigma'_{sp}) = \text{Jord}_{\rho'}(\sigma)$ for $\rho' \not\simeq \rho$ and $\text{Jord}_\rho(\sigma'_{sp}) = \text{Jord}_\rho(\sigma) \setminus \{c_{\min}, c\}$. Using Lemma 2.1, we obtain that there are $a-1 \leq i \leq j \leq b$ and $\pi \otimes \tau \leq \mu^*(\sigma)$ such that

$$\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi$$

and $\sigma'_{sp} \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \tau$. Since $b > (c-1)/2$, it follows $j = b$ and $\pi \simeq \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{-i-1}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho])$. By square-integrability of σ we have two possibilities:

- $i = a - 1$.

This gives $\pi \simeq \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c-1)/2}\rho])$ and Theorem 2.3 of [12] shows that τ is a strongly positive discrete series such that $\text{Jord}(\tau) = \text{Jord}(\sigma) \setminus$

$\{(\rho, c_{\min}), (\rho, c)\}$. It follows that σ'_{sp} is an irreducible subquotient of $\delta([\nu^a \rho, \nu^b \rho]) \rtimes \tau$, contradicting Proposition 3.1 of [11].

- $i = (c_{\min} - 1)/2$.

This gives $\pi \simeq \delta([\nu^{-a+1} \rho, \nu^{(c-1)/2} \rho])$. Using embedding

$$\sigma \hookrightarrow \delta([\nu^{-(c_{\min}-1)/2} \rho, \nu^{(c-1)/2} \rho]) \rtimes \sigma_{sp},$$

where σ_{sp} is strongly positive discrete series such that $Jord(\sigma_{sp}) = Jord(\sigma) \setminus \{(\rho, c_{\min}), (\rho, c)\}$, it directly follows $\tau \leq \delta([\nu^a \rho, \nu^{(c_{\min}-1)/2} \rho]) \rtimes \sigma_{sp}$. This forces

$$\sigma'_{sp} \leq \delta([\nu^{(c_{\min}+1)/2} \rho, \nu^b \rho]) \times \delta([\nu^a \rho, \nu^{(c_{\min}-1)/2} \rho]) \rtimes \sigma_{sp}. \quad (2)$$

Theorem 4.6 of [6] shows $\mu^*(\sigma'_{sp}) \geq \delta([\nu^a \rho, \nu^{(x-1)/2} \rho]) \otimes \sigma''_{sp}$, for $x \in Jord_\rho$ such that $x_- = c$ and strongly positive discrete series σ''_{sp} such that $Jord(\sigma''_{sp}) = Jord(\sigma'_{sp}) \setminus \{(\rho, x)\} \cup \{(\rho, 2a-1)\}$. Applying the structural formula for μ^* on the right-hand side of (2), we obtain that there are $(c_{\min}-1)/2 \leq i_1 \leq j_1 \leq b$, $a-1 \leq i_2 \leq j_2 \leq (c_{\min}-1)/2$ and $\pi' \otimes \sigma' \leq \mu^*(\sigma_{sp})$ such that

$$\begin{aligned} \delta([\nu^a \rho, \nu^{(x-1)/2} \rho]) &\leq \delta([\nu^{-i_1} \rho, \nu^{-(c_{\min}+1)/2} \rho]) \times \delta([\nu^{j_1+1} \rho, \nu^b \rho]) \times \\ &\quad \delta([\nu^{-i_2} \rho, \nu^{-a} \rho]) \times \delta([\nu^{j_2+1} \rho, \nu^{(c_{\min}-1)/2} \rho]) \times \pi' \end{aligned}$$

and

$$\sigma''_{sp} \leq \delta([\nu^{i_1+1} \rho, \nu^{j_1} \rho]) \times \delta([\nu^{i_2+1} \rho, \nu^{j_2} \rho]) \rtimes \sigma'.$$

Since $a > 0, b > (x-1)/2$ and $c_{\min} > 0$, we deduce $i_1 = (c_{\min}-1)/2, j_1 = b$ and $i_2 = a-1$. This obviously gives $\delta([\nu^a \rho, \nu^{(x-1)/2} \rho]) \leq \delta([\nu^{j_2+1} \rho, \nu^{(c_{\min}-1)/2} \rho]) \times \pi'$. Applying [6], Theorem 4.6, we directly obtain $j_2 = a-1$ and $\pi \simeq \delta([\nu^{(c_{\min}+1)/2} \rho, \nu^{(x-1)/2} \rho])$. Using Theorem 4.6 from [6] again, we get that σ' is a strongly positive discrete series such that $Jord(\sigma') = Jord(\sigma_{sp}) \setminus \{(\rho, x)\} \cup \{(\rho, c_{\min})\}$. This forces $\sigma''_{sp} \leq \delta([\nu^a \rho, \nu^b \rho]) \rtimes \sigma'$, contradicting Proposition 3.1 of [11].

Now we treat the case $(c_{\max})_- = c$.

Again we suppose, contrary to our claim, that there is a discrete series subquotient σ_{ds} of $\delta \rtimes \sigma$, attached to the admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$. Again, there has to be some $(d, \rho'') \in Jord^{(1)}$ such that $\epsilon^{(1)}((d_-, \rho''), (d, \rho'')) = 1$. In the same way as in the case $(c_{\max})_- \neq c$ we conclude $\rho'' \simeq \rho$ and d equals either c or c_{\min} . First, if we assume $2a-1 \neq \min(Jord_\rho)$, then $d = c_{\min}$ leads to a contradiction in the same way as in the case. Further, if $2a-1 \neq$

$\min(Jord_\rho)$ and $d = c$, then the properties of ϵ -functions summarized in Section 2 imply that either $\epsilon^{(1)}(((c_{min})_-, \rho), (c_{min}, \rho)) = 1$ or $\epsilon^{(1)}((c, \rho), (x, \rho)) = 1$ holds, where x is an element in $Jord_\rho$ such that $x_- = c$. We have already seen that both these situations are impossible.

It remains to consider the case $d = c$ and $2a - 1 = \min(Jord_\rho)$. Clearly, we can assume $\epsilon^{(1)}(((c_{min}), \rho), (c, \rho)) = 1$ and $\epsilon^{(1)}((c, \rho), (x, \rho)) = -1$ for $x \in Jord_\rho$ such that $x_- = c$. Theorem 8.2 of [17] shows $\sigma_{ds} \hookrightarrow \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma'_{ds}$, for discrete series σ'_{ds} attached to admissible triple $(Jord^{(2)}, \sigma_{cusp}, \epsilon^{(2)})$ such that $Jord^{(2)} = Jord^{(1)} \setminus \{(\rho, c_{min})\} \cup \{(\rho, 2a-1)\}$ and $\epsilon^{(2)}((2a-1, \rho), (c, \rho)) = 1$. Further, from [10] we obtain $\sigma'_{ds} \hookrightarrow \delta([\nu^a \rho, \nu^{(c-1)/2} \rho]) \rtimes \tau$ for tempered subrepresentation τ of $\delta([\nu^{-a+1} \rho, \nu^{a-1} \rho]) \rtimes \sigma'_{sp}$, where σ'_{sp} is a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord^{(2)} \setminus \{(2a-1, \rho), (c, \rho)\}$. This gives

$$\sigma_{ds} \hookrightarrow \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \times \delta([\nu^a \rho, \nu^{(c-1)/2} \rho]) \rtimes \tau.$$

Since the induced representation $\delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \times \delta([\nu^a \rho, \nu^{(c-1)/2} \rho])$ is irreducible, Frobenius reciprocity yields $\mu^*(\sigma_{ds}) \geq \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \times \delta([\nu^a \rho, \nu^{(c-1)/2} \rho]) \otimes \tau$. Thus, the same representation has to be contained in $\mu^*(\delta \rtimes \sigma)$. Since σ is a subrepresentation of $\delta([\nu^{-(c_{min}-1)/2} \rho, \nu^{(c-1)/2} \rho]) \rtimes \sigma_{sp}$ for strongly positive discrete series σ_{sp} such that $Jord(\sigma_{sp}) = Jord \setminus \{(c_{min}, \rho), (c, \rho)\}$, we deduce that there are $a-1 \leq i_1 \leq j_1 \leq b$, $-(c_{min}-1)/2-1 \leq i_2 \leq j_2 \leq (c-1)/2$ and $\pi \otimes \sigma' \leq \mu^*(\sigma_{sp})$ such that

$$\begin{aligned} \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \times \delta([\nu^a \rho, \nu^{(c-1)/2} \rho]) &\leq \delta([\nu^{-i_1} \rho, \nu^{-a} \rho]) \times \delta([\nu^{j_1+1} \rho, \nu^b \rho]) \times \\ &\delta([\nu^{-i_2} \rho, \nu^{(c_{min}-1)/2} \rho]) \times \\ &\delta([\nu^{j_2+1} \rho, \nu^{(c-1)/2} \rho]) \times \pi \end{aligned}$$

and

$$\tau \leq \delta([\nu^{i_1+1} \rho, \nu^{j_1} \rho]) \times \delta([\nu^{i_2+1} \rho, \nu^{j_2} \rho]) \rtimes \sigma'. \quad (3)$$

In a similar manner as before, using $a > 0$, $(c-1)/2 < b$ and Theorem 4.6 from [6], we deduce $i_1 = a-1$, $j_1 = b$, $i_2 = a$ and $j_2 = a-1$. Now we can rewrite (3) as

$$\tau \leq \delta([\nu^a \rho, \nu^b \rho]) \times \delta([\nu^{-a+1} \rho, \nu^{a-1} \rho]) \rtimes \sigma_{sp}. \quad (4)$$

We have already observed $\mu^*(\tau) \geq \delta([\nu^{-a+1} \rho, \nu^{a-1} \rho]) \otimes \sigma'_{sp}$. Repeating the same procedure as before to calculate μ^* of the right-hand side of (4), it may be concluded that σ'_{sp} is an irreducible subquotient of $\delta([\nu^a \rho, \nu^b \rho]) \rtimes$

σ_{sp} , contrary to Proposition 3.1 of [11] since $(c_{max}, \rho) \in \text{Jord}(\sigma_{sp})$ satisfies $2a - 1 < c_{max} < 2b + 1$.

□

Proposition 4.4. *If $\epsilon((2a - 1, \rho), (c_{min}, \rho)) = 1$ and $c_{min} = c_{max}$, then positive generalized principal series contains a discrete series subquotient.*

Proof. To shorten notation, we write $c = c_{min}$. Let σ_{sp} denote a strongly positive discrete series such that $\text{Jord}(\sigma_{sp}) = \text{Jord}(\sigma) \setminus \{(2a - 1, \rho), (c, \rho)\}$. Then σ is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$. By the classification of discrete series, this induced representation has one more discrete series subrepresentation, which we denote by σ' . Similarly, the induced representation $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ contains two discrete series subrepresentations, let us denote them by $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$. For $i \in \{1, 2\}$ we suppose that $\sigma_{ds}^{(i)}$ is attached to admissible triple $(\text{Jord}^{(i)}, \sigma_{cusp}, \epsilon^{(i)})$. Note that $\epsilon^{(i)}((c, \rho), (2b + 1, \rho)) = 1$ for $i \in \{1, 2\}$. We have the following embeddings, for $i \in \{1, 2\}$:

$$\sigma_{ds}^{(i)} \hookrightarrow \delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \rtimes \sigma_{sp} \hookrightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}.$$

By [11], Theorem 2.1, in an appropriate Grothendieck group we have

$$\delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp} = \sigma + \sigma' + L(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}).$$

Let us prove that $\sigma_{ds}^{(i)}$, $i \in \{1, 2\}$, is not a subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp})$. Theorem 8.2 from [17] implies $\mu^*(\sigma_{ds}^{(i)}) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau$, for irreducible representation τ . If we suppose that $\sigma_{ds}^{(i)}$ is a subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp})$, applying Lemma 2.1 and $a > 0$, $(c-1)/2 < b$, we obtain that $\mu^*(L(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp})) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau'$, for some irreducible representation τ' . But, it has been proved in [10] that exactly two representations of such form appear in $\mu^*(\delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ and none of them appears in $\mu^*(L(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}))$, a contradiction.

Thus, $\sigma_{ds}^{(i)}$ is contained in $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma + \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'$, for $i \in \{1, 2\}$. It remains to prove that neither of positive generalized principle series $\delta \rtimes \sigma$ and $\delta \rtimes \sigma'$ can contain both representations $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$. Note that $\mu^*(\sigma_{ds}^{(i)}) \geq \delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \otimes \sigma_{sp}$ for $i \in \{1, 2\}$. Now we calculate multiplicity of $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta \rtimes \sigma)$. There are $a - 1 \leq i \leq j \leq b$ and $\pi \otimes \sigma'' \leq \mu^*(\sigma)$ such that

$$\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \leq \delta([\nu^{-i}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi$$

and

$$\sigma_{sp} \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma''.$$

Since $2b + 1 \notin \text{Jord}_\rho(\sigma)$ and $c = c_{\max}$ it follows that $j \leq (c - 1)/2$. Also, square-integrability of σ gives $i = (c - 1)/2$. Hence $i = j$ and $\pi \simeq \delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho])$. From [10] we directly obtain $\sigma'' \simeq \sigma_{sp}$ and, since the representation $\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{sp}$ appears with multiplicity one in $\mu^*(\sigma)$ we get that the multiplicity of $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta \rtimes \sigma)$ is one.

In completely analogous manner we deduce that the multiplicity of $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta \rtimes \sigma')$ is also one. Therefore, neither of representations $\delta \rtimes \sigma$ and $\delta \rtimes \sigma'$ can contain both representations $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$, so each contains one of them. This ends the proof. \square

Proposition 4.5. *If $\epsilon((2a - 1, \rho), (c_{\min}, \rho)) = 1$ and $c_{\min} = (c_{\max})_-$, then positive generalized principal series contains a discrete series subquotient if and only if $\epsilon((c_{\min}, \rho), (c_{\max}, \rho)) = 1$.*

Proof. As in the proof of the previous proposition, we denote by σ_{sp} the strongly positive discrete series such that $\text{Jord}(\sigma_{sp}) = \text{Jord}(\sigma) \setminus \{(2a - 1, \rho), (c, \rho)\}$ and by σ' other discrete series subrepresentation of the induced representation $\delta([\nu^{-a+1}\rho, \nu^{(c_{\min}-1)/2}\rho]) \rtimes \sigma_{sp}$. We suppose that σ is attached to the admissible triple $(\text{Jord}, \sigma_{\text{cusp}}, \epsilon)$ such that $\epsilon((c_{\min}, \rho), (c_{\max}, \rho)) = 1$. Also, let σ' correspond to the admissible triple $(\text{Jord}', \sigma_{\text{cusp}}, \epsilon')$. Then we have $\epsilon'((c_{\min}, \rho), (c_{\max}, \rho)) = -1$ and $\epsilon(((2a - 1)_-, \rho), (2a - 1, \rho)) = 1$ if $2a - 1 \neq \min(\text{Jord}'_\rho)$. Further, let us denote by σ'_{sp} a strongly positive discrete series such that σ is a subrepresentation of $\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma'_{sp}$ and by σ''_{sp} a strongly positive discrete series such that $\text{Jord}(\sigma''_{sp}) = \text{Jord}(\sigma'_{sp}) \setminus \{(2a - 1, \rho)\} \cup \{(2b + 1, \rho)\}$. Then the induced representation $\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma''_{sp}$ has two irreducible subrepresentation which are both in discrete series. We denote these two representations by $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$ and observe the following embeddings and intertwining operator for $i \in \{1, 2\}$:

$$\begin{aligned} \sigma_{ds}^{(i)} &\hookrightarrow \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma''_{sp} \\ &\hookrightarrow \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'_{sp} \\ &\rightarrow \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma'_{sp}. \end{aligned}$$

Let σ'' stand for discrete series subrepresentation of $\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma'_{sp}$ different than σ . If σ'' is attached to admissible triple $(\text{Jord}, \sigma_{\text{cusp}}, \epsilon'')$,

then $\epsilon((2a-1, \rho), (c_{\min}, \rho)) = -1$. In an appropriate Grothendieck group we have

$$\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma'_{sp} = \sigma + \sigma'' + L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma'_{sp})$$

and it can be seen in the same way as in the proof of previous proposition that $\sigma_{ds}^{(i)}$ is not an irreducible subquotient of $\delta \rtimes L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho])$. Also, Theorem 8.2 of [17] shows that $\mu^*(\sigma_{ds}^{(i)})$ contains $\delta([\nu^a\rho, \nu^{(c_{\min}-1)/2}\rho]) \otimes \tau$ for certain irreducible representation τ and $i \in \{1, 2\}$. Thus, if for some irreducible representation τ' the induced representation $\delta \rtimes \tau'$ contains $\sigma_{ds}^{(i)}$ for some $i \in \{1, 2\}$, Lemma 2.1 and the fact that $(c_{\min} - 1)/2 < b$ imply $\mu^*(\tau') \geq \delta([\nu^a\rho, \nu^{(c_{\min}-1)/2}\rho]) \otimes \tau''$ for some irreducible representation τ'' . Now [17], Proposition 7.2, shows that $\sigma_{ds}^{(i)}$ is not an irreducible subquotient of $\delta \rtimes \sigma''$ for $i \in \{1, 2\}$ and, consequently, both representations $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$ appear in composition series of $\delta \rtimes \sigma$.

It remains to prove that positive generalized principal series $\delta \rtimes \sigma'$ does not contain a discrete series subquotient.

Let us first assume that discrete series $\sigma_{ds}^{(1)}$ is attached to the admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(1)})$ such that $2a-1 \neq \min(Jord_\rho)$ and $\epsilon^{(1)}((2a-1)_-, \rho), (c_{\min}, \rho) = 1$. To simplify notation, we denote $(2a-1)_-$ by x . It follows from [10] that $\mu^*(\sigma_{ds}^{(1)}) \geq \delta([\nu^{(x+1)/2}\rho, \nu^{(c_{\min}-1)/2}\rho]) \otimes \tau_1$, where τ_1 is one of the tempered subrepresentations of $\delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$, where $\sigma_{sp}^{(1)}$ is a strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma''_{sp}) \setminus \{(x, \rho)\} \cup \{(c_{\max}, \rho)\}$. Since $\sigma_{ds}^{(1)}$ is a subquotient of $\delta \rtimes \sigma$, it follows that $\mu^*(\delta \times \delta([\nu^{-a+1}\rho, \nu^{(c_{\min}-1)/2}\rho]) \rtimes \sigma_{sp})$ contains $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{\min}-1)/2}\rho]) \otimes \tau_1$. Using Lemma 2.1, in the same manner as before we deduce that τ_1 is an irreducible tempered subquotient of

$$\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}. \quad (5)$$

Now we calculate the multiplicity of $\delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$ in (5). There are $a-1 \leq i_1 \leq j_1 \leq b$, $-(x-1)/2-1 \leq i_2 \leq j_2 \leq a-1$ and $\pi \otimes \sigma'' \leq \mu^*(\sigma_{sp})$ such that

$$\begin{aligned} \delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) &\leq \delta([\nu^{-i_1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \\ &\quad \delta([\nu^{-i_2}\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{j_2+1}\rho, \nu^{a-1}\rho]) \times \pi \end{aligned}$$

and

$$\sigma_{sp}^{(1)} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{j_2}\rho]) \rtimes \sigma''.$$

Since $(x-1)/2 < a-1$, $a-1 < b$ and σ_{sp} is strongly positive, it follows $i_1 = a-1$, $j_1 = b$, $j_2 = a-1$ and $i_2 = (x-1)/2$. This forces

$$\sigma_{sp}^{(1)} \leq \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}. \quad (6)$$

By Theorem 4.6 from [6], $\mu^*(\sigma_{sp}^{(1)})$ contains $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \otimes \sigma_{sp}''$. Inspecting μ^* of the right-hand side of (6), since representation of the form $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \otimes \sigma'''$ does not appear in $\mu^*(\sigma_{sp})$, we obtain that σ_{sp}'' is contained in $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ and it follows from [11] that it is contained there with multiplicity one.

Therefore, if we denote by τ_2 a tempered subrepresentations of the induced representation $\delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$ different than τ_1 , it follows that τ_2 is not an irreducible subquotient of (5). We recall that τ_1 and τ_2 are not isomorphic.

We also note that $\sigma_{sp}^{(1)}$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{(x+1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ so this induced representation reduces.

Suppose that the proposition were false. Then we could find discrete series $\sigma_{ds}^{(3)}$ which is contained in $\delta \rtimes \sigma'$. We assume that $\sigma_{ds}^{(3)}$ is attached to the admissible triple $(Jord^{(1)}, \sigma_{cusp}, \epsilon^{(3)})$. There is $(d, \rho'') \in Jord^{(1)}$ such that $\epsilon^{(3)}((d_-, \rho''), (d, \rho'')) = 1$. In the same way as in the proof of Proposition 4.3 we deduce $\rho'' \simeq \rho$ and $d \in \{c_{min}, c_{max}, 2b+1\}$. Since we have already seen that $\sigma_{ds}^{(3)} \not\simeq \sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(3)} \not\simeq \sigma_{ds}^{(2)}$, it follows $\epsilon^{(3)}((c_{max}, \rho), (2b+1, \rho)) = \epsilon^{(3)}((c_{min}, \rho), (c_{max}, \rho)) = -1$. The properties of ϵ -function imply $d = c_{min}$ and $x = \min(Jord_\rho)$, for x as before. Since $\mu^*(\sigma_{ds}^{(1)}) \geq \delta([\nu^{(x+1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \tau_2$ and $\sigma' \hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}$, it follows that $\mu^*(\delta \times \delta([\nu^{-a+1}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp})$ also contains the representation $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \tau_2$. It is not hard to see that this can happen only if τ_2 is an irreducible subquotient of (5) and we have already seen that this is impossible. Thus, the proposition is proved. \square

5 Discrete series subrepresentations

In this section we identify a discrete series subrepresentation for every positive generalized principal series which contains a discrete series subquotient.

We start with the most elementary case.

Proposition 5.1. *Let $Jord_\rho \cap [2a+1, 2b+1] = \emptyset$ and denote a discrete series subrepresentation of $\delta \rtimes \sigma$ given by [17], Theorem 8.2, by σ_{ds} . Then in an appropriate Grothendieck group we have*

$$\delta \rtimes \sigma = \sigma_{ds} + L(\delta \rtimes \sigma).$$

Proof. Let us first observe that $L(\delta \rtimes \sigma)$ is a unique non-tempered irreducible subquotient of $\delta \rtimes \sigma$. Suppose, on the contrary, that τ is some non-tempered irreducible subquotient of $\delta \rtimes \sigma$ and write $\tau = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau_t)$, with $\delta_i \simeq \delta([\nu^{a_i} \rho_i, \nu^{b_i} \rho_i])$, $a_i + b_i < 0$, $i = 1, 2, \dots, k$. Similarly as in the proof of Lemma 3.1, we obtain $\mu^*(\tau) \geq \delta_1 \otimes L(\delta_2 \times \cdots \times \delta_k \rtimes \tau_t)$. Using Lemma 2.1, we deduce that there are $a-1 \leq i_1 \leq j_1 \leq b$ and $\pi \otimes \sigma' \leq \mu^*(\sigma)$ such that

$$\delta([\nu^{a_1} \rho_1, \nu^{b_1} \rho_1]) \leq \delta([\nu^{-i_1} \rho, \nu^{-a} \rho]) \times \delta([\nu^{j_1+1} \rho, \nu^b \rho]) \times \pi$$

and

$$L(\delta_2 \times \cdots \times \delta_k \rtimes \tau_t) \leq \delta([\nu^{i_1+1} \rho, \nu^{j_1} \rho]) \rtimes \sigma'.$$

Square-integrability of σ implies $\rho_1 \simeq \rho$, $i_1 \geq a$ and $j_1 = b$. If $b_1 > -a$, we have $\pi \simeq \delta([\nu^{-a+1} \rho, \nu^x \rho])$. This gives $2x+1 \in Jord_\rho$, while square-integrability of σ and assumption $Jord_\rho \cap [2a+1, 2b+1] = \emptyset$ force $x > b$. This is impossible since $i_1 \leq b$ and $-i_1 + x < 0$. Thus, $\delta_1 \simeq \delta([\nu^{-i_1} \rho, \nu^{-a} \rho])$ and $L(\delta_2 \times \cdots \times \delta_k \rtimes \tau_t) \leq \delta([\nu^{i_1+1} \rho, \nu^b \rho]) \rtimes \sigma$. In completely analogous way as in the proof of Lemma 3.1, we deduce $k = 1$. Thus, $\tau_t \leq \delta([\nu^{i_1+1} \rho, \nu^b \rho]) \rtimes \sigma$. Since τ_t has to be discrete series representation, if $i_1 < b$ then $2i_1+1 \in Jord_\rho$. But, since $i_1 \geq a$, this contradicts $Jord_\rho \cap [2a+1, 2b+1] = \emptyset$. Consequently, $\delta_1 \simeq \delta([\nu^{-b} \rho, \nu^{-a} \rho])$ and $\tau_t \simeq \sigma$. This shows $\pi \simeq L(\delta \rtimes \sigma)$.

Now we prove that σ_{ds} is the only tempered subquotient of $\delta \rtimes \sigma$. Obviously, every tempered subquotient of $\delta \rtimes \sigma$ has to be in discrete series, so we denote by σ' a discrete series subquotient of $\delta \rtimes \sigma$ and assume that σ' is attached to the admissible triple $(Jord', \sigma_{cusp}, \epsilon')$. Since σ' is not strongly positive, there is some $(d, \rho') \in Jord'$ such that $\mu^*(\sigma') \geq \delta([\nu^{-(d-1)/2} \rho', \nu^{(d-1)/2} \rho']) \otimes \sigma'_{sp}$, for strongly positive representation σ'_{sp} such that $Jord(\sigma'_{sp}) = Jord' \setminus \{(d, \rho'), (d, \rho')\}$.

Since $Jord_\rho \cap [2a+1, 2b+1] = \emptyset$ it can be easily verified, using Proposition 3.1 of [12] and Lemma 2.1, that representation $\delta([\nu^{-(d-1)/2} \rho', \nu^{(d-1)/2} \rho']) \otimes \sigma'_{sp}$ appears with multiplicity one in $\mu^*(\delta \rtimes \sigma)$. By Theorem 8.2 of [17], such representations also appears in $\mu^*(\sigma_{ds})$ and it follows that $\sigma' \simeq \sigma_{ds}$. This finishes the proof. \square

In the rest of this section we will be concerned with the case $Jord_\rho \cap [2a+1, 2b+1] \neq \emptyset$. As in the previous section, we denote by c_{min} (resp., c_{max}) the minimum (resp., the maximum) of the set $Jord_\rho \cap [2a+1, 2b+1]$. We suppose that $\delta \rtimes \sigma$ has discrete series subquotient and realize σ as an irreducible subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}$, where σ_{sp} is a strongly positive discrete series such that $Jord_\rho(\sigma_{sp}) \cap [2a-1, c_{min}] = \emptyset$.

Proposition 5.2. *Suppose that $\epsilon((2a-1, \rho), (c_{min}, \rho)) = 1$ and $c_{min} = c_{max}$. Then positive generalized principal series contains a discrete series subrepresentation.*

Proof. As in the proof of Proposition 4.4, we write $c = c_{min}$ and denote by σ_{ds} a discrete series subrepresentation of $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ which is contained in $\delta \rtimes \sigma$.

Obviously, we have $\sigma_{ds} \hookrightarrow \delta \times \delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}$ and, by Lemma 3.2 of [10] there is some irreducible representation π such that σ_{ds} is a subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \pi$. We show $\pi \simeq \sigma$.

Let us first show that π has to be a discrete series. Suppose, on the contrary, that π is a non-tempered representation (note that π can not be a tempered representation which is not a discrete series). Using Lemma 3.1, we write π in the standard form $L(\delta_1 \rtimes \tau)$, with $\delta_1 = \delta([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1])$, $a_1 + b_1 < 0$.

This leads to

$$\sigma_{ds} \hookrightarrow \delta \times \delta_1 \rtimes \tau \rightarrow \delta_1 \times \delta \rtimes \tau.$$

Square-integrability of σ_{ds} shows that this representation is contained in the kernel of the last intertwining operator. Thus, $\rho_1 \simeq \rho$ and $a-1 \leq b_1 \leq b-1$. It follows that σ_{ds} is a subrepresentation of

$$\delta([\nu^{a_1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{b_1}\rho]) \rtimes \tau, \quad (7)$$

which is isomorphic to

$$\delta([\nu^a\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^b\rho]) \rtimes \tau. \quad (8)$$

There are two possibilities to consider:

- $b_1 > a-1$.

Now (8) directly implies that $\nu^{b_1}\rho \otimes \tau'$ is an irreducible subquotient of $\mu^*(\sigma_{ds})$ and, by Lemma 3.6 of [10], $(2b_1+1, \rho) \in Jord(\sigma_{ds})$. Since $a_1 + b_1 < 0$ and, by (7), $a_1 + b > 0$, we deduce $2b_1+1 = c$, i.e., $b_1 = (c-1)/2$. Using (7) again we see that $\mu^*(\sigma_{ds}) \geq \delta([\nu^{a_1}\rho, \nu^b\rho]) \otimes \tau''$ for some irreducible representation τ'' and $a_1 < -(c-1)/2$, that is impossible by definition of σ_{ds} .

- $b_1 = a - 1$.

Using Frobenius reciprocity, from (7) we obtain $\mu^*(\sigma_{ds}) \geq \delta([\nu^{a_1}\rho, \nu^b\rho]) \otimes \tau$. Since σ_{ds} is contained in $\delta([\nu^{-(c-1)/2}\rho, \nu^b\rho]) \rtimes \sigma_{sp}$, using Lemma 2.1 we get that there are $-(c+1)/2 \leq i \leq j \leq b$ and $\pi' \otimes \sigma' \leq \mu^*(\sigma_{sp})$ such that

$$\delta([\nu^{a_1}\rho, \nu^b\rho]) \leq \delta([\nu^{-i}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi'$$

and

$$\tau \leq \delta([\nu^i\rho, \nu^{j+1}\rho]) \rtimes \sigma'.$$

In the same way as before, we have that either $i = -a_i$ or $j + 1 = a_1$. Further, it is easy to see that if $\nu^x\rho \otimes \tau'$ appears in $\mu^*(\sigma_{sp})$ then $x < (c-1)/2$ or $b < x$. It follows that τ is a subquotient of $\delta([\nu^{-a_1+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$. Since τ has to be a tempered representation, we conclude $a_1 = -(c-1)/2$ and $\tau \simeq \sigma_{sp}$. In consequence, $\pi \simeq L(\delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$.

On the other hand, using Theorem 8.2 of [17] we deduce $\mu^*(\sigma_{ds}) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma'_{ds}$ for an appropriate discrete series σ'_{ds} . Consequently, $\mu^*(\delta \rtimes \pi) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma'_{ds}$ and $(c-1)/2 < b$ forces $\mu^*(\pi) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ for some irreducible representation π' . Representation $L(\delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ is an irreducible quotient of the induced representation $\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$. It has been proved in [10] that $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp})$ contains exactly two representations of the form $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$, which both appear in Jacquet modules of discrete series subrepresentations of $\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$. Therefore, $\pi \not\leq L(\delta([\nu^{-(c-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$.

Since π is a discrete series, let us denote by $(Jord', \sigma_{cusp}, \epsilon')$ the admissible triple corresponding to π . It is immediate that $Jord' = Jord$. Further, we have already seen $\mu^*(\pi) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$, which implies $\epsilon'((2a-1, \rho), (c, \rho)) = 1$ and π is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$. If $\pi \not\leq \sigma$ then as in the proof of Proposition 4.4 it may be concluded that σ_{ds} is not an irreducible subquotient of $\delta \rtimes \pi$. Therefore, $\pi \simeq \sigma$ and the proof is complete. □

The remaining case is treated in the following proposition.

Proposition 5.3. *Suppose that $\epsilon((2a-1, \rho), (c_{min}, \rho)) = 1$, $c_{min} = (c_{max})-$ and $\epsilon((c_{min}, \rho), (c_{max}, \rho)) = 1$. Then positive generalized principal series contains a discrete series subrepresentation.*

Proof. Let us denote by σ'_{sp} a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(c_{max}, \rho)\} \cup \{(2b+1, \rho)\}$ and by σ''_{sp} a strongly positive discrete series such that $Jord(\sigma''_{sp}) = Jord(\sigma_{sp}) \setminus \{(c_{max}, \rho)\} \cup \{(c_{min}, \rho)\}$. Then the induced representations

$$\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \rtimes \sigma'_{sp} \quad (9)$$

and

$$\delta([\nu^{-(c_{max}-1)/2}\rho, \nu^b\rho]) \rtimes \sigma''_{sp} \quad (10)$$

have a unique common irreducible subrepresentation, which is a discrete series representation and shall be denoted by σ_{ds} . We denote an admissible triple attached to σ_{ds} by $(Jord', \sigma_{cusp}, \epsilon')$.

From (10) it may be concluded that there is some irreducible representation π such that σ_{ds} is a subrepresentation of $\delta \rtimes \pi$. We will show that π is a discrete series representation. Let us first show some results regarding Jacquet modules of π , which will be used afterwards in the proof.

We note that (9) and Frobenius reciprocity imply $\mu^*(\sigma_{ds}) \geq \delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \otimes \sigma'_{sp}$. Since σ_{ds} is a subrepresentation of $\delta \rtimes \pi$, using Lemma 2.1, we deduce that there are $a-1 \leq i \leq j \leq b$ and $\delta' \otimes \pi' \leq \mu^*(\pi)$ such that

$$\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{-a}\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \delta'.$$

From $b > (c_{max} - 1)/2$ follows that $j = b$, so δ has to be equal either $\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c_{max}-1)/2}\rho])$ or $\delta([\nu^{-a+1}\rho, \nu^{(c_{max}-1)/2}\rho])$. In any case, using transitivity of Jacquet modules, we obtain

$$\mu^*(\pi) \geq \delta([\nu^{-a+1}\rho, \nu^{(c_{max}-1)/2}\rho]) \otimes \pi' \quad (11)$$

for some irreducible representation π' . This also allows us to conclude

$$\mu^*(\pi) \geq \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_{max}-1)/2}\rho]) \otimes \sigma' \quad (12)$$

for some irreducible representation σ' .

Further, by [17], Theorem 8.2, we see that there is some irreducible representation σ'' such that $\mu^*(\sigma_{ds}) \geq \delta([\nu^a\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \sigma''$ and in the same way as before we can deduce

$$\mu^*(\pi) \geq \delta([\nu^a\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \pi'' \quad (13)$$

for some irreducible representation π'' .

Suppose, contrary to our assumption, that π is a non-tempered representation. Using Lemma 3.1, we write π in the form $L(\delta_1 \rtimes \tau)$, with $\delta_1 = \delta([\nu^{a_1}\rho_1, \nu^{b_1}\rho_1])$, $a_1 + b_1 < 0$. In the same way as in the proof of the previous proposition, we deduce $\rho_1 \simeq \rho$, $a - 1 \leq b_1 \leq b - 1$ and σ_{ds} is a subrepresentation of

$$\delta([\nu^a\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^b\rho]) \rtimes \tau. \quad (14)$$

Again, there are two possibilities to consider:

- $a > b_1 - 1$.

This forces $(2b_1 + 1, \rho) \in \text{Jord}'$. Thus, $2b_1 + 1$ equals either c_{max} or c_{min} . If $2b_1 + 1 = c_{max}$, then we have $a_1 < -(c_{max} - 1)/2$ and, since $\delta([\nu^a\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^b\rho])$ is isomorphic to $\delta([\nu^{a_1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{b_1}\rho])$, we obtain that there is some irreducible representation σ' such that $\mu^*(\sigma_{ds}) \geq \delta([\nu^{-(c_{max}+1)/2}\rho, \nu^b\rho]) \otimes \sigma'$. But, since in Jord' we have $(2b + 1)_- = c_{max}$, this contradicts Section 3 of [8].

It remains to consider the case $2b_1 + 1 = c_{min}$.

Let us first show $a_1 = -(c_{max} - 1)/2$. Applying the structural formula for μ^* to (10), we obtain that there are $-(c_{max} + 1)/2 \leq i_1 \leq j_1 \leq b$ and $\pi' \otimes \sigma_{sp}^{(1)} \leq \mu^*(\sigma_{sp}'')$ such that

$$\begin{aligned} \delta([\nu^{a_1}\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^{(c_{min}-1)/2}\rho]) &\leq \delta([\nu^{-i_1}\rho, \nu^{(c_{max}-1)/2}\rho]) \times \\ &\delta([\nu^{j_1+1}\rho, \nu^b\rho]) \times \pi' \end{aligned}$$

and

$$\tau \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \rtimes \sigma_{sp}^{(1)}.$$

From definition of σ_{sp}'' , we deduce $\pi' \simeq \delta([\nu^a\rho, \nu^{(c_{min}-1)/2}\rho])$ and $(i_1, j_1) \in \{(-(c_{max} + 1)/2, a_1 - 1), (-a_1, (c_{max} + 1)/2)\}$. Further, this forces

$$\tau \leq \delta([\nu^{-a_1+1}\rho, \nu^{(c_{max}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)},$$

where, by Theorem 4.6 of [6], $\sigma_{sp}^{(1)}$ is a strongly positive discrete series such that $\text{Jord}(\sigma_{sp}^{(1)}) = \text{Jord}(\sigma_{sp}) \setminus \{(c_{max}, \rho)\} \cup \{(2a - 1, \rho)\}$.

Since $a_1 < -(c_{min} - 1)/2$, the representation $\delta([\nu^{-a_1+1}\rho, \nu^{(c_{max}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$ contains a tempered subquotient only if $-a_1 + 1 > (c_{max} - 1)/2$, i.e., for $a_1 = -(c_{max} - 1)/2$. This gives $\pi \simeq L(\delta([\nu^{-(c_{max}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)})$.

It is well known ([10]) that $\mu^*(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)})$ contains exactly two representations of the form $\delta([\nu^{(c_{\min}+1)/2}\rho, \nu^{(c_{\max}-1)/2}\rho]) \otimes \sigma'$ and none of them is contained in $\mu^*(L(\delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^{(c_{\min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}))$. But, this contradicts (12).

- $a = b_1 - 1$.

This clearly forces $\sigma_{ds} \hookrightarrow \delta([\nu^{a_1}\rho, \nu^b\rho]) \rtimes \tau$ and Frobenius reciprocity yields $\mu^*(\sigma_{ds}) \geq \delta([\nu^{a_1}\rho, \nu^b\rho]) \otimes \tau$. Since $a_1 < 0$, Lemma 2.1, applied to (10), implies that the tempered representation τ appears as an irreducible subquotient of the induced representation

$$\delta([\nu^{-a_1+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma_{sp}''.$$

From Proposition 3.1 of [12], we get that either $a_1 = -(c_{\max} - 1)/2$ or $2a_1 + 1 \in \text{Jord}_\rho(\sigma_{sp}'')$ and $\langle 2a_1 + 1, c_{\max} \rangle \cap \text{Jord}_\rho(\sigma_{sp}'') = \emptyset$. It follows directly that $a_1 \in \{-(c_{\max} - 1)/2, -(c_{\min} - 1)/2\}$. Further, if $a_1 = -(c_{\max} - 1)/2$ then $\tau \simeq \sigma_{sp}''$, while $a_1 = -(c_{\min} - 1)/2$ gives $\tau \simeq \sigma_{sp}$.

In the first case we have $\pi \simeq L(\delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}'')$. By Proposition 3.2, in an appropriate Grothendieck group we have

$$\begin{aligned} \delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma_{sp}'' &= \pi + \sigma + L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad L(\delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^{(c_{\min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}). \end{aligned}$$

Let us determine all the representations of the form $\delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \otimes \pi'$ appearing in $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma_{sp}'')$, counting their multiplicities.

From Lemma 2.1 we get that there are $-a \leq i \leq j \leq (c_{\max} - 1)/2$ and $\delta' \otimes \sigma' \leq \mu^*(\sigma_{sp}'')$ such that

$$\delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \times \delta'$$

and

$$\pi' \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'.$$

Obviously, either $-i = -a + 1$ or $j + 1 = -a + 1$. If $j + 1 = -a + 1$, it follows that $\pi' = \sigma_{sp}''$. If $-i = -a + 1$ and $j + 1 = a$, again we obtain $\pi' = \sigma_{sp}''$. It remains to consider the case $-i = -a + 1$ and $j + 1 > a$. Then we have $\delta' \simeq \delta([\nu^a\rho, \nu^j\rho])$ and $2j + 1 \in \text{Jord}_\rho(\sigma_{sp}'')$. Since $a \leq j \leq$

$(c_{max} - 1)/2$, it follows $j = (c_{min} - 1)/2$ and, using Theorem 4.6 of [6], $\pi' \leq \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp}^{(1)}$. Proposition 3.1 of [12] shows that in an appropriate Grothendieck group we have

$$\delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp}^{(1)} = L(\delta([\nu^{-(c_{min}-1)/2} \rho, \nu^{-a} \rho]) \rtimes \sigma_{sp}^{(1)}) + \sigma_{sp}''.$$

Consequently, we have proved that only irreducible representations of the form $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \pi'$ appearing in $\mu^*(\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \rtimes \sigma_{sp}'')$ are $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \sigma_{sp}''$ (which appears with the multiplicity three) and $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2} \rho, \nu^{-a} \rho]) \rtimes \sigma_{sp}^{(1)})$ (which appears with the multiplicity one).

On the other hand, σ and $L(\delta([\nu^{-(c_{min}-1)/2} \rho, \nu^{a-1} \rho]) \rtimes \sigma_{sp})$ are irreducible subquotients of the induced representation $\delta([\nu^{-a+1} \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp}$. Let us also determine all the representations of the form $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \pi'$ appearing in $\mu^*(\delta([\nu^{-a+1} \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp})$. Similarly as before, there are $-a \leq i \leq j \leq (c_{min} - 1)/2$ and $\delta' \otimes \sigma' \leq \mu^*(\sigma_{sp})$ such that

$$\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \leq \delta([\nu^{-i} \rho, \nu^{a-1} \rho]) \times \delta([\nu^{j+1} \rho, \nu^{(c_{min}-1)/2} \rho]) \times \delta'$$

and

$$\pi' \leq \delta([\nu^{i+1} \rho, \nu^j \rho]) \rtimes \sigma'.$$

It is easy to see that either $-i = -a + 1$ or $j + 1 = -a + 1$. If $j + 1 = -a + 1$, then $i = j$, $\delta' \simeq \delta([\nu^{(c_{min}+1)/2} \rho, \nu^{(c_{max}-1)/2} \rho])$ and $\pi' = \sigma_{sp}''$. If $-i = -a + 1$ and $j = a - 1$, again we have $\delta' \simeq \delta([\nu^{(c_{min}+1)/2} \rho, \nu^{(c_{max}-1)/2} \rho])$ and $\pi' = \sigma_{sp}''$. Finally, if $-i = -a + 1$ and $j > a - 1$, using Theorem 4.6 of [6] and description of strongly positive representation σ_{sp} , we deduce $j = (c_{min}-1)/2$, $\delta' \simeq \delta([\nu^a \rho, \nu^{(c_{max}-1)/2} \rho])$ and $\pi' \leq \delta([\nu^a \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp}^{(1)}$.

Thus, the only representations of the form $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \pi'$ appearing in $\mu^*(\delta([\nu^{-a+1} \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp})$ are $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \sigma_{sp}''$ (which appears with the multiplicity three) and $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2} \rho, \nu^{-a} \rho]) \rtimes \sigma_{sp}^{(1)})$ (which appears with the multiplicity one).

Theorem 2.1 from [12] shows that $\delta([\nu^{-a+1} \rho, \nu^{(c_{min}-1)/2} \rho]) \rtimes \sigma_{sp}$ is a length three representation, we denote its discrete series subrepresentation different than σ by σ_1 and its corresponding admissible triple by $(Jord, \sigma_{cusp}, \epsilon_1)$. Since $\sigma \not\simeq \sigma_1$, it follows that $\epsilon \neq \epsilon_1$ and, in consequence, $\epsilon_1((c_{min}, \rho), (c_{max}, \rho)) = -1$. Using Proposition 7.2 of [17] we conclude that $\mu^*(\sigma_1)$ does not contain representation of the form $\delta([\nu^{-a+1} \rho, \nu^{(c_{max}-1)/2} \rho]) \otimes \pi'$.

By Proposition 3.2, representations σ and $L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ are both contained in $\delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \rtimes \sigma''_{sp}$. It follows that $\mu^*(L(\delta([\nu^{-(c_{\max}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma''_{sp}))$ does not contain representation of the form $\delta([\nu^{-a+1}\rho, \nu^{(c_{\max}-1)/2}\rho]) \otimes \pi'$, contradicting (11).

In the second case we have $\pi \simeq L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$. But, we have already observed that $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{(c_{\min}-1)/2}\rho]) \rtimes \sigma_{sp})$ contains exactly two representations of the form $\delta([\nu^a\rho, \nu^{(c_{\min}-1)/2}\rho]) \otimes \pi'$ and none of them is contained in $\mu^*(L(\delta([\nu^{-(c_{\min}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}))$, contrary to (13).

We have now proved that π is a discrete series. Let us denote its corresponding admissible triple by $(Jord'', \sigma_{cusp}, \epsilon'')$. It is evident that $Jord'' = Jord$. Using Proposition 7.2 of [17] and (12), we get $\epsilon''((c_{\min}, \rho), (c_{\max}, \rho)) = 1$, while using Proposition 7.2 of [17] and (13) we get $\epsilon''((2a-1, \rho), (c_{\min}, \rho)) = 1$. Properties of ϵ -function imply that ϵ'' equals -1 on all other pairs. Therefore, $\epsilon'' = \epsilon$ and, in consequence, $\pi \simeq \sigma$. Thus, σ_{ds} is a subrepresentation of $\delta \rtimes \sigma$ and the proposition is proved. \square

We emphasize that it can be easily deduced from the last part of the proof of previous proposition that the other discrete series subquotient of $\delta \rtimes \sigma$, obtained in Proposition 4.5, is not a subrepresentation of $\delta \rtimes \sigma$.

Previous sequence of propositions, together with the results obtained in the previous section, implies the main result of this paper.

Theorem 5.4. *Positive generalized principal series contains a discrete series subquotient if and only if it contains a discrete series subrepresentation.*

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